Writing a Research Paper in Mathematics

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Section 1: Introduction: Why bother?

Good mathematical writing, like good mathematics thinking, is a skill which must be practiced and developed for optimal performance. The purpose of this paper is to provide assistance for young mathematicians writing their first paper. The aim is not only to aid in the development of a well written paper, but also to help students begin to think about mathematical writing.

I am greatly indebted to a wonderful booklet, "How to Write Mathematics," which provided much of the substance of this essay. I will reference many direct quotations, especially from the section written by Paul Halmos, but I suspect that nearly everything idea in this paper has it origin in my reading of the booklet. It is available from the American Mathematical Society, and serious students of mathematical writing should consult this booklet themselves. Most of the other ideas originated in my own frustrations with bad mathematical writing. Although studying mathematics from bad mathematical writing is not the best way to learn good writing, it can provide excellent examples of procedures to be avoided. Thus, one activity of the active mathematical reader is to note the places at which a sample of written mathematics becomes unclear, and to avoid making the same mistakes his own writing.

Mathematical communication, both written and spoken, is the filter through which your mathematical work is viewed. If the creative aspect of mathematics is compared to the act of composing a piece of music, then the art of writing may be viewed as conducting a performance of that same piece. As a mathematician, you have the privilege of conducting a performance of your own composition! Doing a good job of conducting is just as important to the listeners as composing a good piece. If you do mathematics purely for your own pleasure, then there is no reason to write about it. If you hope to share the beauty of the mathematics you have done, then it is not sufficient to simply write; you must strive to write *well*.

This essay will begin with general ideas about mathematical writing. The purpose is to help the student develop an outline for the paper. The next section will describe the difference between "formal" and "informal" parts of a paper, and give guidelines for each one. Section four will discuss the writing of an individual proof. The essay will conclude with a section containing specific recommendations to consider as you write and rewrite the paper.

Section 2. Before you write: Structuring the paper

The purpose of nearly all writing is to communicate. In order to communicate well, you must consider both what you want to communicate, and to whom you hope to communicate it. This is no less true for mathematical writing than for any other form of writing. The primary goal of mathematical writing is to assert, using carefully constructed logical deductions, the truth of a mathematical statement. Careful mathematical readers do not assume that your work is well-founded; they must be convinced. This is your first goal in mathematical writing.

However, convincing the reader of the simple truth of your work is not sufficient. When you write about your own mathematical research, you will have another goal, which includes these two; you want your reader to appreciate the beauty of the mathematics you have done, and to understand its importance. If the whole of mathematics, or even the subfield in which you are working, is thought of as a large painting, then your research will necessarily constitute a relatively minuscule portion of the entire work.

Its beauty is seen not only in the examination of the specific region which you have painted (although this is important), but also by observing the way in which your own work 'fits' in the picture as a whole.

These two goals--to convince your reader of the truth of your deductions, and to allow your audience to see the beauty of your work in relation to the whole of mathematics--will be critical as you develop the outline for your paper. At times you may think of yourself as a travel guide, leading the reader through territory charted only by you.

A successful mathematical writer will lay out for her readers two logical maps, one which displays the connections between her own work and the wide world of mathematics, and another which reveals the internal logical structure of her own work.

In order to advise your reader, you must first consider for yourself where your work is located on the map of mathematics. If your reader has visited nearby regions, then you would like to recall those experiences to his mind, so that he will be better able to understand what you have to add and to connect it to related mathematics. Asking several questions may help you discern the shape and location of your work:

- Does your result strengthen a previous result by giving a more precise characterization of something?
- Have you proved a stronger result of an old theorem by weakening the hypotheses or by strengthening the conclusions?
- Have you proven the equivalence of two definitions?
- Is it a classification theorem of structures which were previously defined but not understood?
- Does is connect two previously unrelated aspects of mathematics?
- Does it apply a new method to an old problem?
- Does it provide a new proof for an old theorem?
- Is it a special case of a larger question?

It is necessary that you explicitly consider this question of placement in the structure of mathematics, because it will linger in your readers' minds until you answer it. Failure to address this very question will leave the reader feeling quite dissatisfied.

In addition to providing a map to help your readers locate your work within the field of mathematics, you must also help them understand the internal organization of your work:

- Are your results concentrated in one dramatic theorem?
- Or do you have several theorems which are related, but equally significant?
- Have you found important counterexamples?
- Is your research purely theoretical mathematics, in the theorem-proof sense, or does your research involve several different types of activity, for example, modeling a problem on the computer, proving a theorem, and then doing physical experiments related to your work?
- Is your work a clear (although small) step toward the solution of a classic problem, or is it a new problem?

Since your reader does not know what you will be proving until after he has read your paper, advising him beforehand about what he will read, just as the travel agent prepares his customer, will allow him to enjoy the trip more, and to understand more of the things you lead him to.

To honestly and deliberately explain where your work fits into the big picture of mathematical research may require a great deal of humility. You will likely despair that your accomplishments seem rather small. Do not fret! Mathematics has been accumulating for thousands of years, based on the work of thousands (or millions) of practitioners. It has been said that even the best mathematicians rarely have more than one really outstanding idea during their lifetimes. It would be truly surprising if yours were to come as a high school student!

Once you have considered the structure and relevance of your research, you are ready to outline your paper. The accepted format for research papers is much less rigidly defined for mathematics than for many other scientific fields. You have the latitude to develop the outline in a way which is appropriate for your work in particular. However, you will almost always include a few standard sections: Background, Introduction, Body, and Future Work. The background will serve to orient your reader, providing the first idea of where you will be leading him. In the background, you will give the most explicit description of the history of your problem, although hints and references may occur elsewhere. The reader hopes to have certain questions answered in this section: Why should he read this paper? What is the point of this paper? Where did this problem come from? What was already known in this field? Why did this author think this question was interesting? If he dislikes partial differential equations, for example, he should be warned early on that he will encounter them. If he isn't familiar with the first concepts of probability, then he should be warned in advance if your paper depends on that understanding. Remember at this point that although you may have spent hundreds of hours working on your problem, your reader wants to have all these questions answered clearly in a matter of minutes.

In the second section of your paper, the introduction, you will begin to lead the reader into your work in particular, zooming in from the big picture towards your specific results. This is the place to introduce the definitions and lemmas which are standard in the field, but which your readers may not know. The body, which will be made up of several sections, contains most of your work. By the time you reach the final section, implications, you may be tired of your problem, but this section is critical to your readers. You, as the world expert on the topic of your paper, are in a unique situation to direct future research in your field. A reader who likes your paper may want to continue work in your field. (S)he will naturally have her/his own questions, but you, having worked on this paper, will know, better than your reader, which questions may be interesting, and which may not. If you were to continue working on this topic, what questions would you ask? Also, for some papers, there may be important implications of your work. If you have worked on a mathematical model of a physical phenomenon, what are the consequences, in the physical world, of your mathematical work? These are the questions which your readers will hope to have answered in the final section of the paper. You should take care not to disappoint them!

Section 3. Formal and Informal Exposition

Once you have a basic outline for your paper, you should consider "the *formal* or *logical* structure consisting of definitions, theorems, and proofs, and the complementary *informal* or *introductory* material consisting of motivations, analogies, examples, and metamathematical explanations. This division of the material should be conspicuously maintained in any mathematical presentation, because the nature of the subject requires above all else that the logical structure be clear." (p.1) These two types of material work in parallel to enable your reader to understand your work both logically and cognitively (which are often quite different--how many of you believed that integrals could be calculated using antiderivatives before you could prove the Fundamental Theorem of Calculus?) "Since the formal structure does not depend on the informal, the author can write up the former in complete detail before adding any of the latter." (p. 2)

Thus, the next stage in the writing process may be to develop an outline of the logical structure of your paper. Several questions may help: To begin, what exactly have you proven? What are the lemmas (your own or others) on which these theorems stand. Which are the corollaries of these theorems? In deciding which results to call lemmas, which theorems, and which corollaries, ask yourself which are the central ideas. Which ones follow naturally from others, and which ones are the real work horses of the paper? The structure of writing requires that your hypotheses and deductions must conform to a linear order. However, few research papers actually have a linear structure, in which lemmas become more and more complicated, one on top of another, until one theorem is proven, followed by a sequence of increasingly complex corollaries. On the contrary, most proofs could be modeled with very complicated graphs, in which several basic hypotheses combine with a few well known theorems in a complex way. There may be several seemingly independent lines of reasoning which converge at the final step. It goes without saying that any assertion should follow the lemmas and theorems on which it depends. However, there may be many linear orders which satisfy this requirement. In view of this difficulty, it is your responsibility to, first, understand this structure, and, second, to arrange the necessarily linear structure of your writing to reflect the structure of the work as well as possible. The exact way in which this will proceed depends, of course, on the specific situation.

One technique to assist you in revealing the complex logical structure of your paper is a proper naming of results. By naming your results appropriately (lemmas as underpinnings, theorems as the real substance, and corollaries as the finishing work), you will create a certain sense of parallelness among your lemmas, and help your reader to appreciate, without having struggled through the research with you, which are the really critical ideas, and which they can skim through more quickly.

Another technique for developing a concise logical outline stems from a warning by Paul Halmos, in HTWM, never to repeat a proof:

> If several steps in the proof of Theorem 2 bear a very close resemblance to parts of the proof of Theorem 1, that's a signal that something may be less than completely understood. Other symptoms of the same disease are: 'by the same technique (or method, or device, or trick) as in the proof of Theorem 1...', or, brutally, 'see the proof of Theorem 1'. When that happens the chances are very good that there is a lemma that is worth finding, formulating, and proving, a lemma from which both Theorem 1 and Theorem 2 are more easily and more clearly deduced. (p. 35)

These issues of structure should be well thought through BEFORE you begin to write your paper, although the process of writing itself which surely help you better understand the structure.

Now that we have discussed the formal structure, we turn to the informal structure. The formal structure contains the formal definitions, theorem-proof format, and rigorous logic which is the language of 'pure' mathematics. The informal structure complements the formal and runs in parallel. It uses less rigorous, (but no less accurate!) language, and plays an important part in elucidating both the mathematical location of the work, as we discussed above, and in presenting to the reader a more cognitive presentation of the work. For although mathematicians *write* in the language of logic, very few actually *think* in the language of logic (although we do think logically), and so to understand your work, they will be immensely aided by subtle demonstration of *why* something is true, and how you came to prove such a theorem. Outlining, before you write, what you hope to communicate in these informal sections will, most likely, lead to more effective communication.

Before you begin to write, you must also consider notation. The selection of notation is a critical part of writing a research paper. In effect, you are inventing a language which your readers must learn in order to understand your paper. Good notation firstly allows the reader to forget that he is learning a new language, and secondly provides a framework in which the essentials of your proof are clearly understood. Bad notation, on the other hand, is disastrous and may deter the reader from even reading your paper. In most cases, it is wise to follow convention. Using epsilon for a prime integer, or x(f) for a function, is certainly possible, but almost never a good idea.

Section 4: Writing a Proof

The first step in writing a good proof comes with the statement of the theorem. A well-worded theorem will make writing the proof much easier. The statement of the theorem should, first of all, contain exactly the right hypotheses. Of course, all the necessary hypotheses must be included. On the other hand, extraneous assumptions will simply distract from the point of the theorem, and should be eliminated when possible.

When writing a proof, as when writing an entire paper, you must put down, in a linear order, a set of hypotheses and deductions which are probably not linear in form. I suggest that, before you write you map out the hypotheses and the deductions, and attempt to order the statements in a way which will cause the least confusion to the reader.

In HTWM, Halmos offers several important recommendations about writing proofs:

1. Write the proof forward

> A familiar trick of bad teaching is to begin a proof by saying: "Given e, let d be e/2". This is the traditional backward proof-writing of classical analysis. It has the advantage of being easily *verifiable* by a machine (as opposed to *understandable* by a human being), and it has the dubious advantage that something at the end comes out to be less than e. The way to make the human reader's task less demanding is obvious: write the proof forward. Start, as the author always starts, by putting something less than e, and then do what needs to be done--multiply by 3M2 + 7 at the right time and divide by 24 later, etc., etc.--till you end up with what you end up with. Neither arrangement is elegant, but the forward one is graspable and rememberable. (p. 43)

2. Avoid unnecessary notation. Consider:

> a proof that consists of a long chain of expressions separated by equal signs. Such a proof is easy to write. The author starts from the first equation, makes a natural substitution to get the second, collects terms, permutes, inserts and immediately cancels an inspired factor, and by steps such as these proceeds till he gets the last equation. This is, once again, coding, and the reader is forced not only to learn as he goes, but, at the same time, to decode as he goes. The double effort is needless. By spending another ten minutes writing a carefully worded paragraph, the author can save each of his readers half an hour and a lot of confusion. The paragraph should be a recipe for action, to replace the unhelpful code that merely reports the results of the act and leaves the reader to guess how they were obtained. The paragraph would say something like this: "For the proof, first substitute p for q, the collect terms, permute the factors, and, finally, insert and cancel a factor r. (p. 42-43)

Section 5. Specific Recommendations

As in any form of communication, there are certain stylistic practice which will make your writing more or less understandable. These may be best checked and corrected after writing the first draft. Many of these ideas are from HTWM, and are more fully justified there.

- Notation that hasn't been used in several pages (or even paragraphs) should carry a reference or a reminder of the meaning.
- The structure should be easily discernible by headings and punctuation.
- There should be a clear definition of the problem at hand all the way through.
- The title is the first contact that readers will have with your paper. It must communicate something of the substances to the experts in your field as well as to the novices who will be interested. Thus, while the terminology should be technically correct

- "Don't over work a small punctuation mark such as a period or comma. They are easy for the reader to overlook, and the oversight causes backtracking, confusion, delay. Example: "Assume that aÂX. X belongs to the class C, ...". The period between the two X's is overworked... A good general rule is: never start a sentence with a symbol. If you insist on starting the sentence with a mention of the thing the symbol denotes, put the appropriate word in apposition, thus: "The set X belongs to the class C, ..."...The overworked period is no worse than the overworked comma. Not "For invertible X, X* also is invertible", but "For invertible X, the adjoint X* also is invertible." (p. 44)
- "I recommend "then" with "if" in all mathematical contexts. The presence of "then" can never confuse; its absence can."(p.44)
- Another critical feature is the layout or architecture of the page. "If it looks like solid prose, it will have a forbidding, sermony aspect; if it looks like computational hash, with a page full of symbols, it will have a frightening, complicated aspect. The golden mean is golden. Break it up, but not too small; use prose, but not too much. Intersperse enough displays to give the eye a chance to help the brain; use symbols, but in the middle of enough prose to keep the mind from drowning in a morass of suffixes." (p. 44-5)
- The same symbol should never be used for more than one thing; if you have used n as a counter in one proof, use m in the next proof, unless the two play a similar role in each
- All notation should be meaningful (no free variables):
- "avoid the use of irrelevant symbols. Example: "On a compact space every real-valued continuous function *f* is bounded." What does the symbol "f" contribute to the clarity of that statement?... A showy way to say "use no superfluous letters" is to say "use no letter only once". (p. 41)
- "In everyday English "any" is an ambiguous word; depending on context that may hint at an existential quantifier ("have you any wool?", "if anyone can do it, he can") or a universal one ("any number can play"). Conclusion: never use "any" in mathematical writing. Replace it by "each" or "every", or recast the whole sentence" (p. 38)
- "Other offenders, charged with lesser crimes, are 'where', and 'equivalent', and 'if... then...if...then'. "Where" is usually a sign of a lazy afterthought that should have been thought through before. "If *n* is sufficiently large, then |*an*| < e, where e is a preassigned positive number"; both disease and cure are clear. "Equivalent" *for theorems* is logical nonsense. (By "theorem" I mean a mathematical truth, something that has been proved. A meaningful statement can be false, but a theorem cannot; "a false theorem" is self-contradictory). As for "if...then...if...then", that is just a frequent stylistic bobble committed by quick writers and rued by slow readers. "If *p*, then if *q*, then *r*." Logically, all is well, but psychologically it is just another pebble to stumble over, unnecessarily. Usually all that is needed to avoid it is to recast the sentence, but no universally good recasting exists; what is best depends on what is important in the case at hand. It could be "If *p* and *q* then *r*", or "In the presence of *p*, the hypothesis *q* implies the conclusion *r*", or many other versions."" (p. 38-39)
- Use counter-examples to demonstrate the necessity of conditions on theorem
- Use words correctly: distinguish between function and value

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